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## Uniqueness of the asymptotic $AdS_3$ geometry

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### Abstract

We explicitly show that in  $(2+1)$  dimensions the general solution of the Einstein equations with negative cosmological constant on a neighbourhood of timelike spatial infinity can be obtained from BTZ metrics by coordinate transformations corresponding geometrically to deformations of their spatial infinity surface. Thus, whatever the topology and geometry of the bulk, the metric on the timelike extremities is BTZ.

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Graham and Lee [1] proved that, under suitable topological assumptions, Euclidean Einstein spaces with negative cosmological constant  $\Lambda$  are completely defined by the geometry on their boundary. Furthermore, Fefferman and Graham (FG) [2] showed that, whatever the signature, there exists an asymptotic expansion of the metric, which formally solves the Einstein equations with  $\Lambda = -1/\ell^2 < 0$ ; we choose hereafter the length units such that  $\ell = 1$ . The first terms of this expansion may be given by even powers of a radial coordinate  $r$  :

$$ds^2 \underset{\bar{r} \rightarrow \infty}{\simeq} dr^2 r^2 + r^2 \overset{(0)}{\mathbf{g}}(x^i) + \overset{(2)}{\mathbf{g}}(x^i) + \dots \quad . \quad (1)$$

On  $(n+1)$ -dimensional space-times, the full asymptotic expansion continues with terms of negative even powers of  $r$  up to  $r^{-2(\lfloor \frac{n+1}{2} \rfloor - 2)}$ , with in addition a logarithmic term of the order of  $r^{-(n-2)} \log r$  when  $n$  is even and larger than

2. All these terms are completely defined by the boundary geometry  $\overset{(0)}{\mathbf{g}}(x^i)$ , assumed to be non-degenerate (i.e.  $n$ -dimensional and Lorentzian). They are

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followed by terms of all negative powers starting from  $r^{-(n-2)}$ . The trace-free part of the  $r^{-(n-2)}$  coefficient is not fully determined by  $\mathbf{g}^{(0)}$ ; it contains degrees of freedom in Lorentzian spaces [3, 4]. Once this ambiguity is fixed, all subsequent terms become determined.

On (2+1)-dimensional spaces, Bañados [5] showed, for flat boundary geometries, that the FG expansion stops at order  $r^{-2}$  on a neighbourhood  $\mathcal{V}_\infty \simeq \mathcal{I} \times \Sigma_\infty$ , where  $\Sigma_\infty$  is a timelike component of the surface at infinity and  $\mathcal{I}$  is a semi-infinite open interval  $]r_0, \infty[$  of the  $r$ -variable; such a neighbourhood is hereafter called an extremity. The most general asymptotic solution<sup>3</sup> he obtains is of the form :

$$ds^2 = \frac{dr^2}{r^2} + (rdx^+ + \frac{1}{r}\mathcal{L}_-(x^-)dx^-)(rdx^- + \frac{1}{r}\mathcal{L}_+(x^+)dx^+) \quad . \quad (2)$$

This metric depends on two functions of one variable,  $\mathcal{L}_+(x^+)$  and  $\mathcal{L}_-(x^-)$ . Skenderis and Solodukhin [6] generalized this result by showing that in (n+1) dimensions, for any locally AdS metric, i.e. any conformally flat metric with Riemann tensor  $R^{\mu\nu}_{\rho\sigma} = \delta^\mu_\sigma \delta^\nu_\rho - \delta^\mu_\rho \delta^\nu_\sigma$ , the FG expansion can be written as :

$$ds^2 = \frac{dr^2}{r^2} + \left( r\delta_i^k + \frac{1}{2r} g^{(2)}_{il} g^{(0)lk} \right) g^{(0)}_{km} \left( r\delta_j^m + \frac{1}{2r} g^{(0)mp} g^{(2)}_{pj} \right) dx^i dx^j \quad , \quad (3)$$

where  $\mathbf{g}^{(0)}$  is a conformally flat boundary metric [7].

In this letter, we explicit the link between the  $AdS_3$  metrics (2, 3), the BTZ black hole metrics [8] characterized by their mass  $M$  and angular momentum  $J$ , and the Liouville fields [9, 7, 6] that encode the boundary degrees of freedom, under the assumption that the surface at infinity is topologically a cylinder :  $\Sigma_\infty \simeq \mathbb{R} \times S^1$ . Since solutions with different values of  $M$  and  $J$  are distinct, the question arises whether or not solutions in which the constants  $M$  and  $J$  are replaced by functions are more general. Our starting point is the observation that the coordinates  $(r, x^i)$  constitute a Gaussian coordinate system (see ref. [10], chap. 3) with respect to the family of surfaces  $r = cst$ ; the boundary corresponds to  $r = \infty$ . Accordingly, we shall examine Gaussian coordinate systems built on families of surfaces, embedded in BTZ spaces, which asymptotically merge into the boundaries of these spaces.

The BTZ black hole geometry is characterized by two parameters, the mass  $M$  and the angular momentum  $J$ ; for simplicity we restrict ourselves first to the non-extremal case  $M > |J|$ . Its metric reads as :

$$ds^2 = -(\rho^2 - M)dt^2 + \frac{d\rho^2}{\rho^2 - M + \frac{J^2}{4\rho^2}} + \rho^2 d\varphi^2 + Jd\varphi dt \quad . \quad (4)$$

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<sup>3</sup>By asymptotic solution we mean an exact solution of the Einstein equations with negative cosmological constant, which is valid on  $\mathcal{V}_\infty$ , but which does not necessarily admit a singularity-free extension.

Posing  $M_{\pm} = \frac{1}{4}(M \pm J)$  and performing the coordinate transformation :

$$\xi^2 = \frac{\rho^2 - M_+ - M_- + \rho \sqrt{\rho^2 - 2(M_+ + M_-) + \frac{1}{\rho^2}(M_+ - M_-)^2}}{2\sqrt{M_+ M_-}} \quad , \quad (5)$$

$$y^{\pm} = \sqrt{M_{\pm}}(\varphi \pm t) \quad , \quad (6)$$

this metric can be written in the FG form :

$$ds^2 = \frac{d\xi^2}{\xi^2} + (\xi dy^+ + \frac{1}{\xi} dy^-)(\xi dy^- + \frac{1}{\xi} dy^+) \quad . \quad (7)$$

This form reflects the local equivalence of all the BTZ metrics. Their global differences are encoded in the periods of the variables, with the identification  $y^{\pm} + 2\pi\sqrt{M_{\pm}} \equiv y^{\pm}$ .

The metric (7) admits two Killing vectors  $\partial_{y^{\pm}}$ . This allows to integrate by quadratures the geodesic equations in these coordinates. Denoting by  $C_{\pm}$  the (constant) values of the moments conjugate to  $y^{\pm}$ , by  $\gamma$  an affine parameter on these geodesics, and by  $\gamma_0$  and  $z^{\pm}$  integration constants, the spacelike geodesics can be written as :

$$\gamma - \gamma_0 = \log[\Gamma(\xi, C_+, C_-)] - \log[\Gamma(\xi_0, C_+, C_-)] \quad , \quad (8)$$

$$y^{\pm} - z^{\pm} = Y^{\pm}(\xi, C_+, C_-) - Y^{\pm}(\xi_0, C_+, C_-) \quad , \quad (9)$$

where we have introduced the primitives expressed in terms of  $X = \xi^2 + \xi^{-2}$  :

$$\begin{aligned} \log[\Gamma(\xi, C_+, C_-)] &= \int \frac{\xi^4 - 1}{\sqrt{(\xi^4 - 1)^2 - 4\xi^2(C_- \xi^2 - C_+)(C_+ \xi^2 - C_-)}} \frac{d\xi}{\xi} \\ &= \frac{1}{2} \log \frac{1}{2} [X - 2C_+ C_- + \sqrt{X^2 - 4C_+ C_- X + 4(C_+^2 + C_-^2 - 1)}] \quad , \quad (10) \end{aligned}$$

$$\begin{aligned} Y^{\pm}(\xi, C_+, C_-) &= 2 \int \frac{\xi(C_{\mp} - 2C_{\pm}\xi^2 + C_{\mp}\xi^4)}{(\xi^4 - 1)\sqrt{(\xi^4 - 1)^2 - 4\xi^2(C_- \xi^2 - C_+)(C_+ \xi^2 - C_-)}} d\xi \\ &= \frac{1}{4} \left\{ \log \frac{(2+X)(C_+ C_- - C_+ - C_- + 1)}{(2-X)(C_+ C_- \mp C_+ \pm C_- - 1)} \right. \\ &\quad \left. + \log \frac{2(C_+ C_- - C_+^2 - C_-^2 + 1) - (1 - C_+ C_-)X \pm (C_- - C_+)\sqrt{X^2 - 4C_+ C_- X + 4(C_+^2 + C_-^2 - 1)}}{2(C_+ C_- + C_+^2 + C_-^2 - 1) - (1 + C_+ C_-)X + (C_- + C_+)\sqrt{X^2 - 4C_+ C_- X + 4(C_+^2 + C_-^2 - 1)}} \right\} \quad . \quad (11) \end{aligned}$$

Consider now the family of surfaces  $\Sigma(\rho_0)$  :

$$\xi e^{-\Delta(y^+, y^-)} = \rho_0 \quad , \quad (12)$$

where the function  $\Delta$  has the periodicities  $2\pi\sqrt{M_{\pm}}$  of  $y^+$  and  $y^-$ . In the limit where  $\rho_0$  tends to  $\infty$ , the geodesic congruence, orthogonal to  $\Sigma(\rho_0)$  at the point

of coordinates  $y^\pm = z^\pm$ ,  $\xi = \xi_0 \equiv \rho_0 \exp[\Delta(z^+, z^-)]$ , is obtained by fixing the constants  $C_+$  and  $C_-$  as :

$$C_\pm = \partial_{y^\pm} \Delta|_{y^\pm=z^\pm} \equiv C_\pm(z^+, z^-) \quad . \quad (13)$$

If we moreover impose, for all the geodesics of this congruence, the value  $\gamma_0$  of  $\gamma$  on  $\Sigma(\rho_0)$  to be equal to  $\log \rho_0$ , we obtain in the same limit  $\rho_0 \rightarrow \infty$  a finite expression of  $\gamma$  as a function of  $\xi$  that reads as :

$$\gamma = \Delta(z^+, z^-) + \log[\Gamma(\xi, C_+, C_-)] \quad , \quad (14)$$

which asymptotically leads to :

$$\gamma = \Delta(z^+, z^-) + \log \xi - C_+ C_- \xi^{-2} + \frac{1}{2}(C_-^2 + C_+^2 - 3C_-^2 C_+^2) \xi^{-4} + O(\xi^{-6}) \quad . \quad (15)$$

Posing  $r = \exp(\gamma)$  and defining  $\Gamma_*$  as the inverse function of  $\Gamma$  and  $Z^\pm$  as the composed functions  $Y^\pm[\Gamma_*(r e^\Delta, C_+, C_-), C_+, C_-]$ , we obtain :

$$\xi = \Gamma_*[r e^\Delta, C_+, C_-] \quad (16)$$

$$\simeq r e^\Delta \left( 1 + C_+ C_- \frac{1}{(r e^\Delta)^2} - \frac{1}{2}(C_+^2 + C_-^2) \frac{1}{(r e^\Delta)^4} \right) + O(r^{-5}) \quad (17)$$

$$y^\pm = z^\pm + Z^\pm[r e^\Delta, C_+, C_-] \quad (18)$$

$$\simeq z^\pm - C_\mp \frac{1}{(r e^\Delta)^2} + C_\pm (1 + C_\mp^2) \frac{1}{(r e^\Delta)^4} + O(r^{-6}) \quad , \quad (19)$$

In agreement with the general statement of [6], the FG expansion stops at order  $r^{-2}$ , whatever the (conformally flat) boundary geometry is. Accordingly, to obtain the expression of the metric in terms of the new variables  $r$  and  $z^\pm$ , it suffices to use eqs (17, 19); contributions of subsequent terms in the asymptotic expansions of  $\xi$  and  $y^\pm$  must indeed cancel out. Hence, the metric (7) becomes :

$$\begin{aligned} ds^2 = & \frac{dr^2}{r^2} + [1 + \frac{e^{-2\Delta}}{r^2} \partial_{z^+ z^-}^2 \Delta][1 - (\partial_{z^+} \Delta)^2 + \partial_{z^+}^2 \Delta] (dz^+)^2 \\ & + [1 + \frac{e^{-2\Delta}}{r^2} \partial_{z^+ z^-}^2 \Delta][1 - (\partial_{z^-} \Delta)^2 + \partial_{z^-}^2 \Delta] (dz^-)^2 \\ & + \{ r^2 e^{2\Delta} + 2 \partial_{z^+ z^-}^2 \Delta \\ & + \frac{e^{-2\Delta}}{r^2} [(\partial_{z^+ z^-}^2 \Delta)^2 + [1 - (\partial_{z^+} \Delta)^2 + \partial_{z^+}^2 \Delta][1 - (\partial_{z^-} \Delta)^2 + \partial_{z^-}^2 \Delta]] \} dz^+ dz^- \quad (20) \end{aligned}$$

We would like to stress that this expression is not a truncated asymptotic approximation, but an exact solution of the Einstein equations, valid in a neighbourhood of the surface at infinity (a tedious calculation confirms that the expected cancellations occur and that the Ricci tensor is :  $R_\mu^\nu = -2\delta_\mu^\nu$ ).

Now, let us split the deformation function as :

$$\Delta(z^+, z^-) = \mathcal{D}(z^+, z^-) + \frac{1}{2}(\mathcal{A}_+(z^+) + \mathcal{A}_-(z^-)) \quad ; \quad (21)$$

this form becomes unambiguous once we fix for instance  $\mathcal{D}(z^+, 0) = 0$  and  $\mathcal{D}(0, z^-) = 0$ . We then perform the changes of variables :

$$x^\pm = \int e^{\mathcal{A}_\pm(z^\pm)} dz^\pm \quad , \quad (22)$$

and rewrite the metric as :

$$ds^2 = \frac{dr^2}{r^2} + \left[ re^D dx^+ + \frac{1}{re^D} (L_- dx^- + (\partial_{x^+ x^-}^2 D) dx^+) \right] \left[ re^D dx^- + \frac{1}{re^D} (L_+ dx^+ + (\partial_{x^+ x^-}^2 D) dx^-) \right] \quad , \quad (23)$$

where :

$$L_\pm(x^+, x^-) = e^{-2A_\pm} - (\partial_{x^\pm} D)^2 + \partial_{x^\pm}^2 D + \frac{1}{4}(\partial_{x^\pm} A_\pm)^2 + \frac{1}{2}\partial_{x^\pm}^2 A_\pm \quad , \quad (24)$$

and  $D(x^+, x^-) = \mathcal{D}[z^+(x^+), z^-(x^-)]$ ,  $A_\pm(x^\pm) = \mathcal{A}_\pm[z^\pm(x^\pm)]$ .

Considering the terms of order  $r^2$  and  $r^0$  in the metric (23), we recognize the equation obtained in [4, 7, 6, 11] :

$$g_{\mu\nu}^{(2)} = \frac{1}{2}[T_{\mu\nu}(\phi^L) - g_{\mu\nu}^{(0)} \mathcal{R}] \quad , \quad (25)$$

where  $\mathcal{R}$  is the 2-dimensional Gauss curvature of the asymptotic geometry

$\mathbf{g}^{(0)} = e^{2D(x^+, x^-)} dx^+ dx^-$ , and  $T_{\mu\nu}[\phi^L]$  the energy-momentum tensor of the Liouville field defined as :

$$\phi^L(x^+, x^-) = \phi_0^L(x^+, x^-) - 2D(x^+, x^-) \quad , \quad (26)$$

where  $\phi_0^L$  is directly obtained from the Liouville field associated to the BTZ metric (7) re-expressed in  $x^\pm$  coordinates :

$$\phi_0^L(x^+, x^-) = \log \frac{|\partial_{x^+} f_+ \partial_{x^-} f_-|}{(f_+ + f_-)^2} \quad \text{with} \quad f_\pm(x^\pm) = \frac{a_\pm e^{2z^\pm(x^\pm)} + b_\pm}{c_\pm e^{2z^\pm(x^\pm)} + d_\pm} \quad (27)$$

depending on the parameters  $a_\pm, b_\pm, c_\pm, d_\pm$  such that  $a_\pm d_\pm - b_\pm c_\pm = 1$  (see ref. [7]).

To be well-defined, the functions  $\mathcal{D}(z^+, z^-)$  and  $\mathcal{A}_\pm(z^\pm)$  must be periodic in  $z^+$  and  $z^-$  :

$$\mathcal{D}(z^+ + 2\pi\sqrt{M_+}, z^- + 2\pi\sqrt{M_-}) = \mathcal{D}(z^+, z^-) \quad , \quad (28)$$

$$\mathcal{A}_\pm(z^\pm + 2\pi\sqrt{M_\pm}) = \mathcal{A}_\pm(z^\pm) \quad . \quad (29)$$

Hence, we immediately see that the periods  $P_\pm$  of the variables  $x^\pm$  are given by :

$$P_\pm = \int_0^{2\pi\sqrt{M_\pm}} e^{\mathcal{A}_\pm(z^\pm)} dz^\pm \quad ; \quad (30)$$

they are, in general, different from those of the initial variables  $y^\pm$ . Thus, if  $M$  and  $J$  are given, this equation yields directly the periodicity of the variables  $x^\pm$  and of the functions appearing in eq. (23). Conversely, if we start from the expression (23) of the metric, the functions  $D$  and  $L_\pm$  allow to determine the periodicities  $P_\pm$  of the  $x^\pm$  variables and, as a consequence, the values of  $M$  and  $J$  via de equation :

$$2\pi\sqrt{M_\pm} = \int_0^{P_\pm} e^{-A_\pm(x^\pm)} dx^\pm \quad . \quad (31)$$

These considerations clarify the previously noted [7] relationship between the eigenvalues of Wilson loop matrices, the Floquet exponents occurring in the solutions of the parallel transport equations, and the mass and angular momentum of the BTZ black holes, in agreement with the fact that eigenvalues of holonomy matrices corresponding to loops of constant  $\xi$  represent an invariant concept with respect to coordinate transformations.

Let us conclude with a few remarks. First we would like to stress that the metric (23), obtained from the BTZ metric (7) by a change of coordinates, corresponds to the most general form of the FG asymptotic expansion (1) in (2+1) dimensions. Indeed, as all two dimensional geometries with cylindrical topology are conformally flat, any boundary geometry can be written as  $\mathbf{g}^{(0)} = e^{2D(x^+, x^-)} dx^+ dx^-$ . Moreover,  $\mathbf{g}^{(2)} = L_+(dx^+)^2 + L_-(dx^-)^2 + 2(\partial_{x^+ x^-}^2 D) dx^+ dx^-$  constitutes the most general subdominant metric satisfying the Einstein equations and, according to the FG theorem, all subsequent terms  $\mathbf{g}^{(2n)}$  with  $n \geq 2$  are unambiguously fixed by  $\mathbf{g}^{(0)}$  and  $\mathbf{g}^{(2)}$ . This provides an alternative proof in (2+1) dimensions that the most general metric expansion stops at order  $r^{-2}$  [6]. The metric (2) is recovered from this general metric by imposing a flat boundary geometry.

We showed hereabove the construction of the FG expansion, starting from non-extremal BTZ black hole metrics (7). To consider more general cases, it is sufficient to start from the following generalization of (7) :

$$ds^2 = \frac{d\xi^2}{\xi^2} + (\xi dy^+ + \frac{\epsilon_-}{\xi} dy^-)(\xi dy^- + \frac{\epsilon_+}{\xi} dy^+) \quad , \quad (32)$$

where the  $\epsilon_\pm$  are the “signs of  $M_\pm$ ”. This means that for non-extremal BTZ black hole metrics  $(\epsilon_+, \epsilon_-) = (1, 1)$ , for extremal ones  $(\epsilon_+, \epsilon_-)$  is  $(1, 0)$  or  $(0, 1)$  when  $M = |J| \neq 0$  and  $(0, 0)$  if  $M = |J| = 0$ ;  $(\epsilon_+, \epsilon_-) = (1, -1)$  [resp.  $(-1, 1)$ ] with  $y^-$  [resp.  $y^+$ ] non-periodic corresponds to self [resp. anti self]-dual solutions [12],  $(\epsilon_+, \epsilon_-) = (-1, -1)$  with both  $y^\pm$  non-periodic to pure AdS geometry, and the remaining possibilities with periodic  $y^+$  and/or  $y^-$  to spaces with closed time lines. For all these cases, we obtain the same final expression (23) of the metric, but with the functions  $L_\pm$  now defined as :

$$L_\pm(x^+, x^-) = \epsilon_\pm e^{-2A_\pm} - (\partial_{x^\pm} D)^2 + \partial_{x^\pm}^2 D + \frac{1}{4}(\partial_{x^\pm} A_\pm)^2 + \frac{1}{2}\partial_{x^\pm}^2 A_\pm \quad . \quad (33)$$

Furthermore, given an FG expression (23) of the metric, the functions  $A_{\pm}(x^{\pm})$  can be computed from  $D(x^+, x^-)$  and  $L_{\pm}(x^+, x^-)$  using eq.(33). If the so obtained functions  $\exp[A_{\pm}(x^{\pm})]$  are positive, the change of variables (22) is well-defined and the metric's singularities are just coordinate singularities corresponding to the locus of points where neighbouring geodesics intersect. If the functions  $A_{\pm}(x^{\pm})$  are moreover periodic, the starting FG metric is simply a local description of some asymptotic BTZ geometry, corresponding to a deformation of the surface at infinity defined by the standard coordinates  $\rho$ ,  $\varphi$  and  $t$ ; the values of the mass and angular momentum are encoded in the periodicity of the  $z^{\pm}$  variables, or in a more hidden way in the periodicity of the  $x^{\pm}$  coordinates via the functions  $D(x^+, x^-)$  and  $L_{\pm}(x^+, x^-)$ . On the other hand, if the functions  $A_{\pm}(x^{\pm})$  are not periodic (i.e. for  $P_{\pm} = 0$ ), of course no identifications are allowed and the metric is nothing else than a local metric on a pure AdS space [8]. However, we may also consider metrics (23) constructed with functions  $L_{\pm}$  that cannot be obtained from (7) or (32) by a change of coordinates, in particular when  $A_{\pm}$  becomes singular and  $e^{-A_{\pm}}$  negative. In such case, the metric (23) still solves the Einstein equations, but may present true singularities, whose interpretation requires further investigation. They may for instance be the sign of the presence of multi-black holes in the bulk of the space [13, 14].

Let us emphasize the significance of our results. Of course, for fixed  $\Lambda$ , all  $AdS_3$  metrics are locally isometric, whatever the arbitrary  $x^{\pm}$ -dependent functions occurring in them (see eq. (32)). What we show here is that for any such functions the metric can be transformed on a whole extremity  $\mathcal{V}_{\infty}$  of the space into a canonical BTZ expression with specified values of  $M$  and  $J$ , via eq. (31). The degrees of freedom that give rise to the black hole entropy are thus not more encoded in these functions than in  $M$  and  $J$ . It is however still conceivable that they are encoded through the existence of more complex topologies, with several disconnected components of the spatial infinity surface. Finally note that the coordinate transformations (8, 9) are nothing else than the finite group transformations whose infinitesimal generators provide the well known Virasoro algebra of asymptotic symmetries [15].

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